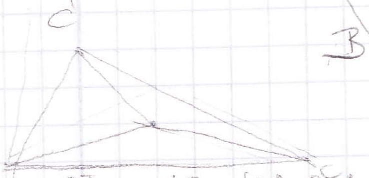
22.19b) ex. (Fermat or Toricelli point, Napoleon's theorem):

Let ABC be a Δ with angles $< \frac{2}{3}\pi$, and C', A', B' the outside vertices of equilateral triangles over AB, BC, AC.

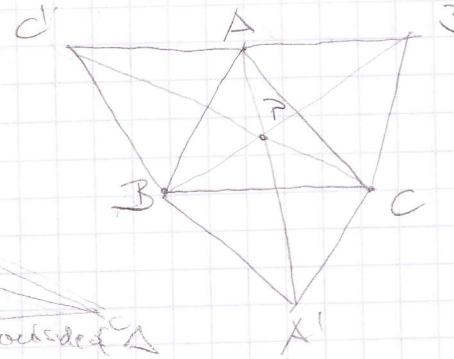
Then $A'A$, $B'B$, $C'C$ are concurrent (intersect) in

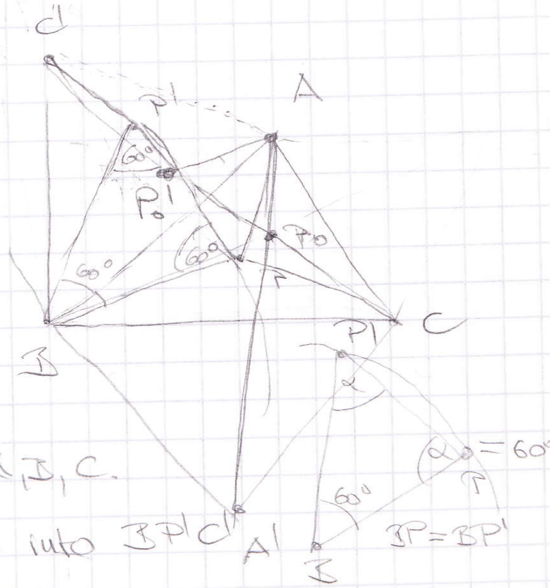
$$P = \underset{\{A,B,C\}}{\arg\min}\ 1/\mathbb{R}^2/\cdot/\ell_2/\Sigma = \text{med}\ \Delta ABC$$

P is called Fermat or Toricelli or 1st isogonic point.



$\sphericalangle > \frac{2\pi}{3} \Rightarrow C'C$ outside of Δ





Proof (Hofmann [1929]):

i) Select $P \in \Delta ABC$, connect P to A, B, C.

Rotate ΔBPA $\frac{2\pi}{6}$ ($60°$) around B into $BP'C'$

$\Rightarrow \Delta BPP'$ is equilateral $\Rightarrow PB = P'P$

$PA = P'C' = C'P'$

$\Rightarrow PA + PB + PC = C'P' + P'P + PC \geq C'C$

C' (as the image of A) does not depend on P,

BAC' is equilateral.

Construct equilateral $\Delta BA'C$

Let $P^0 = CC' \cap AA'$

$\Delta C'CB$ is a $60°$-rotation of $\Delta AA'B$ around B

$\Rightarrow$ The $60°$ rotation P_0' of P_0 around B lies on $C'C$

$\Rightarrow PA + PB + PC = C'P_0' + P_0'P_0 + P_0C = C'C$ is minimal

Moreover for $P \neq P_0$ $P_0' \notin CC'$

$\Rightarrow P_0 = \underset{P \in \Delta}{\arg\min}\ PA + PB + PC$

and AA', BB', CC' intersect in P_0

(Napoleon's theorem).

ii) Select $P \notin \triangle ABC$.

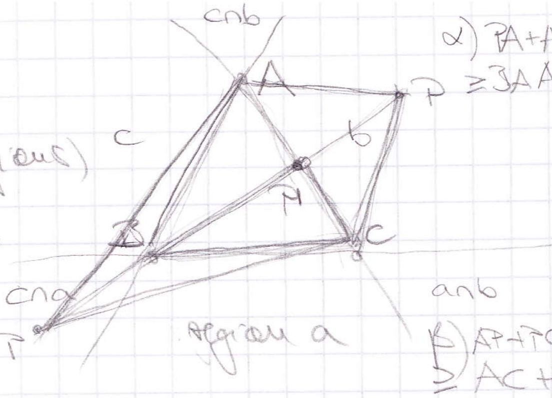
α) w.l.o.g. $P \in c \cap a$ (two regions)

$\Rightarrow PA + PB + PC > BA + BB + BC$

β) w.l.o.g. $P \in b$ (one region)

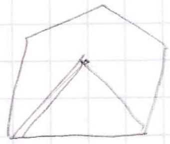
Let $P' = AC \cap PB$

$\Rightarrow PA + PB + PC > P'A + P'B + P'C$



α) $PA + AC + PC (+PB) \geq BA + AC + BC$

β) $AP + PC + CB + BA \geq AC + BC + BA$



$u(\Delta) \leq u(\text{Polygon})$

22.20 Example $(1/\mathbb{R}^1/-/\ell_1/\mathbb{Z}_{\to w_i})$:

$\{a_i\}_{i=1}^{25} = \{2, 5, 8, 18, 21, 15, 15, 15, 15, 15, 15, 19, 19, 19, 21, 19, 20, 23, 25, 28, 5, 13, 25, 28, 31\}$

Sort a_i as

$$(2, \underbrace{5, 5}_{2\times}, 8, 13, \underbrace{15, 15, 15, 15, 15, 15}_{6\times}, 18, \overset{13}{\downarrow}\underbrace{19, 19, 19, 19}_{4\times}, 20, \underbrace{21, 21}_{2\times}, 23, \underbrace{25, 25}_{2\times}, \underbrace{28, 28}_{2\times}, 31)$$

$\Rightarrow (b_i) = (2, 5, 8, 13, 15, 18, 19, 20, 21, 23, 25, 28, 31)$ $\Rightarrow \text{med}\{a_i\} = \min_{13}\{a_i\} = 19$

$(w_i) = (1, 2, 1, 1, 6, 1, 4, 1, 2, 1, 2, 2, 1)$

$\Rightarrow \text{med}_{w_i}\{b_i\} = 19$.

Run time is $O(m \log m)$ because of the sort. Best possible?

22.21 Algorithm (select $(k, a_1, \ldots, a_m)$)

Input: $a \in \mathbb{R}^m$, $1 \le k \le m$

Output: $\min_k \{a_i\}_{i=1}^m$

Data structures: $(p_j) \in \mathbb{R}^{m/5}$ (ignoring integrality issues), $p \in \mathbb{R}$

1. for $j = 1, \ldots, \frac{m}{5}$ $\quad O(m/5)$

   $p_j \leftarrow \text{select}(3, a_{(j-1)\cdot 5}, \ldots, a_{(j-1)5+4})$ // medians of every 5 elements $\quad O(m/5)$

2. $p \leftarrow \text{select}(\lceil \frac{m}{10} \rceil, p_1, \ldots, p_{\frac{m}{5}})$ $\quad T(\frac{m}{5})$

3. $L \leftarrow \{a_i : a_i < p\}$

   $E \leftarrow \{a_i : a_i = p\}$ $\quad O(m)$

   $G \leftarrow \{a_i : a_i > p\}$

4. if $|L| \ge k$ then

   $p \leftarrow \text{select}(k, L)$ $\quad T(\frac{7}{10}m)$

   elseif $|L| + |E| < k$ then $\quad T(\frac{7}{10}m)$

   $p \leftarrow \text{select}(k - |L| - |E|, G)$ $\quad T(\frac{7}{10}m)$

   endif

5. return p

22.22 Theorem (Blum, Floyd, Pratt, Rivest, Tarjan [1972])

Let $T(m,k) := \sup_{s = \langle k, a_1, \ldots, a_m \rangle} f_{\text{select}}(s)$, $T(m) := \max_{k=1}^{m} T(m,k)$

$T(n) = O(n)$, i.e., Select runs in linear time.
i.e., the k-th min of n numbers can be computed in linear time.

Proof:

i) $|L+E|, |G+E| \geq \frac{3}{10} n \Rightarrow |L|, |G| \leq \frac{7}{10} n$

W.l.o.g. let $p_1 \leq \dots \leq p_{\frac{n}{5}}$. Then

$$|L \cup E| \geq \Big| \underbrace{\bigcup_{i=1}^{\lceil \frac{n}{5} \cdot \frac{1}{2} \rceil} \bigcup_{j=1}^{3} \{\min_j \underbrace{\{a_{(i-1)5}, \dots, a_{(i-1)5+4}\}}_{5 \text{ numbers}}\}}_{3 \text{ (out of 5) numbers are } \leq \text{ the median}} \Big| \geq \left\lceil \frac{n}{5} \cdot \frac{1}{2} \right\rceil \cdot 3 \geq \frac{3}{10} n$$

Analogously $|G \cup E| \geq 3/10\, n$.

ii) $T(n) = cn + T(\frac{n}{5}) + T(\frac{7}{10} n)$

$= cn + T(\frac{2}{10} n) + T(\frac{7}{10} n)$

"stack of bricks method"

$= \boxed{cn}$

$+ \boxed{\frac{2}{10} cn} \boxed{\frac{7}{10} cn}$ $\qquad \frac{9}{10} cn$

$+ \boxed{}\boxed{}\boxed{}\boxed{}$ $\qquad \frac{81}{100} cn$

$+ \dots$

$\leq cn \sum_{i=0}^{\infty} \left(\frac{9}{10}\right)^i$

$= \frac{1}{1-\frac{9}{10}} cn = 10\, cn = O(n)$.

iii) In general (i.e., in particular for $5 \nmid n$):

α) $|L \cup E| \geq \left(\left\lceil \frac{n}{5} \right\rceil - 1\right) \frac{1}{2} \cdot 3 \geq \frac{3}{10} n - \frac{3}{2} \geq \frac{3}{10} n (1-\varepsilon)$

$\Leftrightarrow \frac{3}{10} n \cdot \varepsilon \geq \frac{3}{2} \Leftrightarrow n \geq \frac{5}{\varepsilon}$, analogously $|G| + |E| \geq \frac{3}{10}(1-\varepsilon) n$

$\Rightarrow |L|, |G| \leq \frac{10}{10} n - \frac{3}{10} n (1-\varepsilon) = \frac{10-3+3\varepsilon}{10} n = \frac{7+3\varepsilon}{10} n$

β) $\left\lceil \frac{n}{5} \right\rceil \leq \frac{n}{5} + 1 \leq \frac{n}{5}(1+\varepsilon) \Leftrightarrow 1 \leq \frac{n\varepsilon}{5} \Leftrightarrow \frac{5}{\varepsilon} \leq n$

γ) $T(n) = cn + T\left(\frac{n}{5}(1+\varepsilon)\right) + T\left(\frac{7+3\varepsilon}{10} n\right)$

$= cn + T\left(\frac{2+2\varepsilon}{10} n\right) + T\left(\frac{7+3\varepsilon}{10} n\right)$

$\leq cn \cdot \sum_{i=0}^{\infty} \Big(\underbrace{\frac{2+2\varepsilon+7+3\varepsilon}{10}}\Big)^i$

$= cn \frac{1}{1-\delta}$ $\qquad \frac{9+5\varepsilon}{10} =: \delta < 1 \Leftrightarrow \varepsilon < \frac{1}{5}$

$= O(n)$ for $n \geq \frac{5}{\varepsilon} > \frac{5}{1/5} = 25$

$\Rightarrow$ solve Problem for $n \leq 25$ (by sorting) in constant time.

$\square$

22.23 Theorem: Let $T:\mathbb{N}_0 \to \mathbb{N}_0$ satisfy the recursion

$$T(n) = c_0 n + \sum_{i=1}^{k} T(\lceil c_i n \rceil), \quad c_i \geq 0,\ i=0,\dots,k; \quad \sum_{i=1}^{k} c_i < 1.$$

Then $T(n) = O(n)$.

Proof: Stack of bricks + rounding. □

22.20 Ex (cont'd): Select$\left(\frac{25}{2}, 2, \dots, 31\right)$

22.24 Algorithm qselect$(k, a_1, \dots, a_n)$ (Quickselect)

Input: $a \in \mathbb{R}^n$, $1 \leq k \leq n$

Output: $\min_k \{a_i\}_{i=1}^{n}$

Data structures: $p \in \mathbb{R}^n$, ℓ (global)

1. $p \leftarrow$ randomly select a_i, $i=1,\dots,n$, $\ell \leftarrow 1$ $\quad O(1)$
2. relabel $a_1 \leq \dots < a_i = p = \dots = a_j < \dots \leq a_n$ $\quad O(n-1)$ comparisons
3. If $i=k$ then $p \leftarrow a_i$ $\quad O(1)$
   If $i>k$ then $p \leftarrow$ qselect$(k, a_1, \dots, a_{i-1})$
   If $i<k$ then $p \leftarrow$ qselect$(k-i, a_{i+1}, \dots, a_n)$ $\quad \leq \frac{\tilde{T}(0) + \tilde{T}(1) + \dots + \tilde{T}(n-1)}{n}$ max
4. $\ell \leftarrow \ell+1$ $\quad O(1)$
5. return p. $\quad O(1)$

$\sum_{i=0}^{n-1} T(i)$

$\sum_{i=1}^{n} T(i-1)$

$\sum_{i=1}^{n} 4i - \sum_{i=1}^{n} 4$

$\left[\frac{n(n+1)}{2} - n\right] \cdot 4$

$\frac{n(n-1)}{2} \cdot 4$

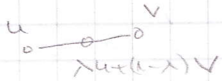
22.25 Thm: Let $\tilde{T}(n,k) = E\left[f_{qselect}(s = \langle k, a_1, \dots, a_n \rangle)\right]$, $\tilde{T}(n) := \max_{k=1}^{n} \tilde{T}(n,k)$. Then $\tilde{T}(n) = O(n)$, i.e., quickselect runs in expected linear time.

Proof: $\tilde{T}(n) \leq \overbrace{2(n-1)}^{\leq n-1+4} + \frac{1}{n} \sum_{i=0}^{n-1} \tilde{T}(i)$. Assume $\tilde{T}(i) \leq 4i$ for $i<n$.

$$\leq 2(n-1) + \frac{4}{\cancel{n}} \frac{(n-1)\cancel{n}}{2}$$

$$\leq 4(n-1) \leq 4n. \quad \square$$

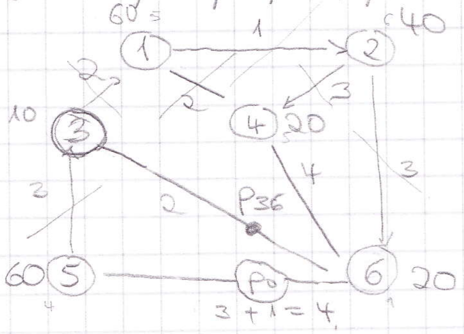
2.3 Medians in Networks



2.3.1. Ex. (Schöbel & Schmidt [2009]):

$\in V \cup A \cup E$ — shortest path distance w.r.t. N, $sp(\lambda u+(1-\lambda)v, w) := \min\{(1-\lambda)c_{uv}+sp(u,w),\ \lambda c_{uv}+sp(v,w)\}$

Warehouse: $1 / N / \cdot / sp / \Sigma$



undirected directed edges

$N = (V, E \cup A)$ network

$c_e, c_a \in \mathbb{R}_+$ edge weights

i) $p=4$: $\sum_{v\in V} sp(4,v) = 2+3+6+0+8+4 = 23$

ii) $p=p_0$: $\sum_{v\in V} sp(p_0,v) = 5+6+3+5+3+1 = 23$

2.3.2. Thm. (Node solution, Hakimi [1964]): There is $v\in V$ s.t.

$v \in \operatorname{argmin} 1/N/\cdot/sp/\Sigma$,

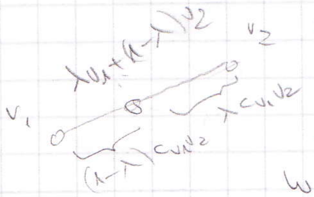
i.e., the 1-median problem on a network has an optimal node solution.

Proof: Consider $p = \lambda v_1 + (1-\lambda)v_2$, $\lambda\in(0,1)$ for:

a) $v_1v_2 \in A$: $\sum_{v\in V} sp(p,v) \geq \sum_{v\in V} sp(v_2,v)$.

b) $v_1v_2 \in E$: $\sum_{v\in V} sp(p,v) = \sum_{v\in V} \min\{\underbrace{(1-\lambda)c_{v_1v_2} + sp(v_1,v)}_{:\Leftrightarrow v\in V_1},\ \underbrace{\lambda c_{v_1v_2} + sp(v_2,v)}_{:\Leftrightarrow v\in V_2}\}$

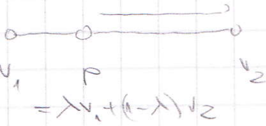
$$= \sum_{v\in V_1}\left[(1-\lambda)c_{v_1v_2} + sp(v_1,v)\right] + \sum_{v\in V_2}\left[\lambda c_{v_1v_2} + sp(v_2,v)\right]$$



$$= c_{v_1v_2}\left[(1-\lambda)|V_1| + \lambda|V_2|\right] + \sum_{v\in V_1} sp(v_1,v) + \sum_{v\in V_2} sp(v_2,v)$$

$$\underset{|V_1|\leq|V_2|}{\overset{\text{w.l.o.g.}}{\geq}} c_{v_1v_2}\left[1\cdot|V_1| + 0\,|V_2|\right] + \sum_{v\in V_1} sp(v_1,v) + \sum_{v\in V_2} sp(v_2,v)$$

$$\overset{(p\to v_2)}{=} \sum_{v\in V_1}\underbrace{\left[c_{v_1v_2} + sp(v_1,v)\right]}_{\geq sp(v_2,v)} + \sum_{v\in V_2} sp(v_2,v)$$



$$\geq \sum_{v\in V} sp(v_2,v).$$ □

2.3.3. Alg. (for $1/N/\cdot/sp/\Sigma$)

Input: $N=(V, E\cup A)$, $c_{uv}\in\mathbb{R}_+$, $uv \in E\cup A$

Output: p^* − argmin $1/N/\cdot/sp/\Sigma$

1. compute shortest path matrix $(sp(u,v))_{uv\in V^2}$ (e.g., Floyd-Warshall)
2. compute row sums $sp^T \mathbb{1}$
3. output index of row sum minimum $p^* = \operatorname{argmin} sp^T \mathbb{1}$. □

2.3.1 ex (cont'd):

$$Sp = \begin{array}{c|cccccc|c} & 1 & 2 & 3 & 4 & 5 & 6 & \\ \hline 1 & 0 & 1 & 6 & 2 & 8 & 4 & 21 \\ 2 & 5 & 0 & 5 & 3 & 7 & 3 & 23 \\ 3 & 2 & 3 & 0 & 4 & 6 & 2 & 17 \\ 4 & 2 & 3 & 6 & 0 & 8 & 4 & 23 \\ 5 & 5 & 6 & 3 & 7 & 0 & 4 & 25 \\ 6 & 4 & 5 & 2 & 4 & 4 & 0 & 19 \end{array} \Rightarrow p^* = 3.$$

2.3.4. ex (weighted median in a network):

Volumes w_i : $1/N|\cdot/sp/\sum w_i$

$w_i = (60, 40, 10, 20, 60, 20) \Rightarrow p^* = \text{argmin } sp^T w.$

$$\begin{pmatrix} 0 & 1 & 6 & 2 & 8 & 4 \\ 5 & 0 & 5 & 3 & 7 & 3 \\ 2 & 3 & 0 & 4 & 6 & 2 \\ 2 & 3 & 6 & 0 & 8 & 4 \\ 5 & 6 & 3 & 7 & 0 & 4 \\ 4 & 5 & 2 & 4 & 4 & 0 \end{pmatrix} \begin{pmatrix} 60 \\ 40 \\ 10 \\ 20 \\ 60 \\ 20 \end{pmatrix} = \begin{pmatrix} 700 \\ 890 \\ 720 \\ 260 \\ 790 \\ 780 \end{pmatrix} \Rightarrow p^* = 1.$$

2.4 Center in Networks

2.3.4 ex. (cont'd): Fire brigade $1/N|\cdot/sp/\max$

2.4.1. ex. (Edge selection):

$\text{argmin } 1/N|\cdot/sp/\max = \frac{1}{2} v_1 + \frac{1}{2} v_2.$

2.4.2 Obs: Center problems in networks do not necessarily have optimal node selections allowing only node selections

2.4.3. Alg. (for $1/N|N|\cdot/sp/\max$)

Input: $N = (V, E \cup A)$, $c_{uv} \in \mathbb{R}_+$, $uv \in E \cup$

Output: $p^* = \text{argmin } 1/N|V|\cdot/sp/\max$

1. compute $(sp(u,v))_{uv \in V^2}$
2. compute row maxima $\max\limits_{v \in V} (sp(u,v))$
3. output index of minimal row max. $p^* = \text{argmin} \max\limits_{v} sp(u,v)$ □

2.3.4. ex. (cont'd):

$$\begin{pmatrix} 0 & 1 & 6 & 2 & 8 & 4 \\ 5 & 0 & 5 & 3 & 7 & 3 \\ 2 & 3 & 0 & 4 & 6 & 2 \\ 2 & 3 & 6 & 0 & 8 & 4 \\ 5 & 6 & 3 & 7 & 0 & 4 \\ 4 & 5 & 2 & 4 & 4 & 0 \end{pmatrix} \quad \begin{matrix} 8 \\ 7 \\ 6 \\ 8 \\ 7 \\ 5 \end{matrix} \Rightarrow p^* = 6, \quad \max_{v \in V} sp(6,v) = 5$$

2.4.4 Lem. (Preprocessing): Let $p = \lambda v_1 + (1-\lambda) v_2$, $\lambda \in (0,1)$, $v_1v_2 \in E \cup A$ and $\bar{p} = \arg\min 1/N|V|\cdot/sp/\max$. Then $p \notin \arg\min 1/N|\cdot/sp/\max$ if

a) $v_1v_2 \in A$ and $c_{v_1v_2} > 0$

b) $v_1v_2 \in E$ and $sp(v_1,\bar{v}), sp(v_2,\bar{v}) > \max_{v\in V} sp(\bar{p},v)$ for some $\bar{v} \in V$.

Proof:

a) $\max_{v\in V} sp(p,v) > \max_{v \in V} sp(v_2,v)$.

b) $\max_{v\in V} sp(p,v) \geq sp(p,\bar{v}) \geq \min_{i=1,2} sp(v_i,\bar{v}) > \max_{v\in V} sp(\bar{p},v)$. □

2.3.4 Ex (cont'd): $14 \in E$, $p \in \mathring{14}$

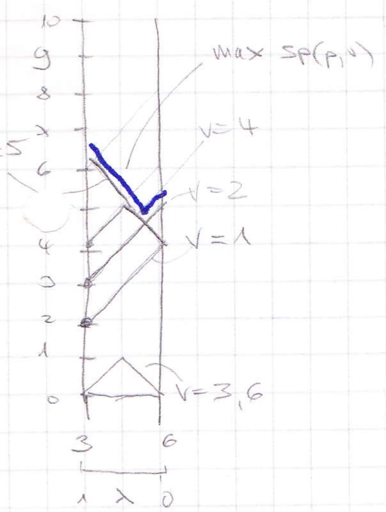
$sp(p,5) \geq \min\{sp(1,5), sp(4,5)\} = \min\{8,8\} = 8 \geq 5 = \max_{v\in V} sp(\overset{\bar{p}}{\overset{\shortparallel}{6}},v)$

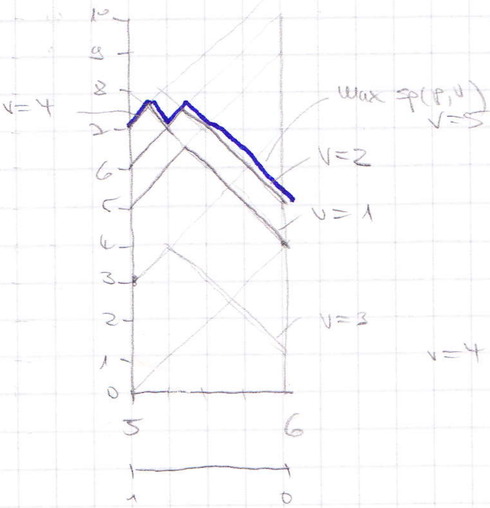
2.4.5 Obs. (Undirected edges): Let $p = \lambda v_1 + (1-\lambda) v_2$, $\lambda \in (0,1)$, $v_1v_2 \in E$

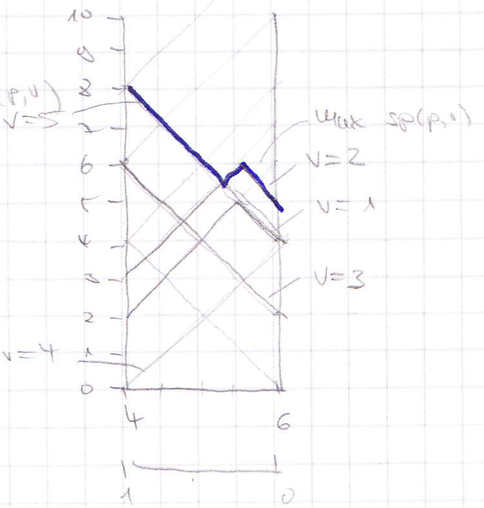
a) $sp(p,v) = \min\{(1-\lambda)c_{v_1v_2} + sp(v_1,v), \lambda c_{v_1v_2} + sp(v_2,v)\}$, $v \in V$
is a piecewise affine, continuous function in λ with ≤ 2 pieces

b) $\max sp(p,v)$ is a piecewise affine, continuous function in λ
with $\leq 2|V|$ pieces.

2.3.4 Ex (cont'd)







$\Rightarrow \lambda = 0{,}25$ $\quad \rightarrow \lambda = 0$ $\quad \Rightarrow \lambda = 0$

$\Rightarrow p_{36} = \frac{1}{4}(3) + \frac{3}{4}(6)$ $\quad \rightarrow p_{56} = (6)$ $\quad \Rightarrow p_{46} = 6$

$\max_{v\in V} sp(p_{36},v) = \frac{9}{2}$ $\quad \max_{v\in V} sp(6,v) = 5$ $\quad \max_{v\in V} sp(6,v) = 5$

$6 = \arg\min 1/N|V/\cdot/sp/\max$

$\Rightarrow p^* = \arg\min 1/N|\cdot/sp/\max = \frac{1}{4}(3) + \frac{3}{4}(6) = p_{36}$.

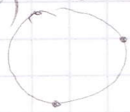
21.05.12

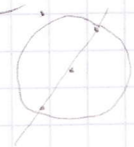
2.7 Centers in the Plane

2.7.1 Proposition (Smallest enclosing circle): Let $V = \{v_i\}_{i=1}^m \subseteq \mathbb{R}^2$.

a) There is a unique circle C of smallest diameter enclosing V.

b) $|C \cap V| \geq 2$ and $\underbrace{|C \cap V|}_{=\{v_1, v_2\}} = 2 \Rightarrow \text{dist}(v_1, v_2) = \text{diam } C$.





Proof: Ex. □

2.7.2 Algorithm (Elzinga & Hearn [1971]):

Input: $v_1, \dots, v_m \in \mathbb{R}^2$

Output: $p^* \in \text{argmin } 1/\mathbb{R}^2/\cdot/l_2/\max$

Data structures: a, b, c, d $p^* \in \mathbb{R}^2$

1. $a, b \leftarrow V$, $V \leftarrow V \setminus \{a, b\}$
2. $C \leftarrow$ circle with diameter a, b, center $p^* = \frac{1}{2}a + \frac{1}{2}b$

   If $\text{conv } C \supseteq V$ then goto 5

   else $c \leftarrow V \setminus \text{conv } C$, $V \leftarrow V \setminus \{c\}$

   endif
3. If Δabc has right or obtuse angle then

   relabel a, b, c s.t. c is at right or obtuse angle

   drop c, goto 2

   endif
4. $C \leftarrow$ smallest enclosing circle of Δabc, center p^*

   If $\text{conv } C \supseteq V$ then goto 5

   else $d \leftarrow V \setminus \text{conv } C$, $V \leftarrow V \setminus \{d\}$

   relabel a, b, c s.t. $\|a - d\| \geq \|b - d\|, \|c - d\|$

   construct line L_{ap^*} through a, p^*

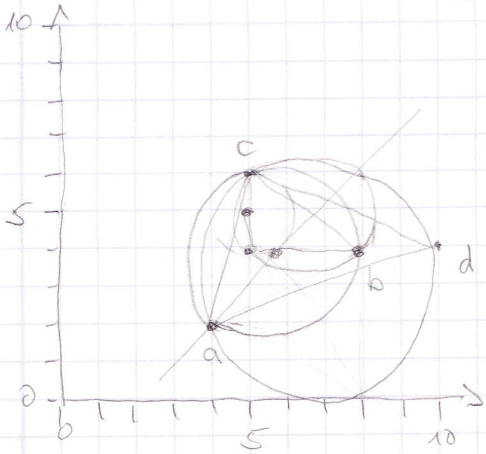
   relabel b, c s.t. b, d are on the same side of L_{ap^*}

   $b \leftarrow d$

   goto 3
5. output p^*

2.7.3 Ex. $V = \{(5,4), (5,6), (5,5), (8,4), (4,2), (10,4)\}$

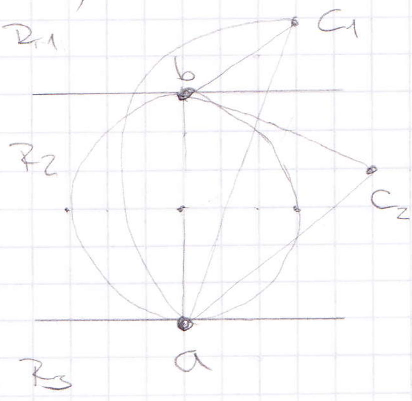
$(5,4) \to a$, $(5,6) \to b$



2.7.4 <u>Thm:</u> Alg. 2.7.2 is correct.

<u>Proof:</u> The algorithm produces circles C defined by 2 points in step 2 and by 3 points in step 4. We show: C has strictly increasing diameter. Then the alg. terminates finitely with $C \supseteq \operatorname{conv} V$, as there are only a finite number of 2- and 3-point circles.

a) Let C be defined by 2 points a, b. Consider regions R_1, R_2, R_3 defined by lines through a, b orthogonal to ab.



i) $\max \measuredangle \Delta abc \geq \frac{\pi}{2} \Leftrightarrow c \in R_1 \cup R_3$, say $c \in R_1$

next circle is defined by ac and $ac > ab$.

ii) $\max \measuredangle \Delta abc < \frac{\pi}{2}$

next circle is defined by Δabc and has larger diameter because of Prop 2.7.1.

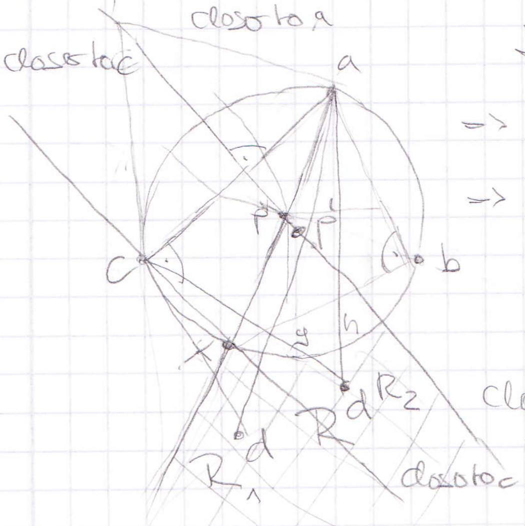
b) Let C be defined by 3 points forming acute Δabc.

Let x be the opposite of a on C

$\Rightarrow b \in \overset{\frown}{ax}$, $c \in \overset{\frown}{xa}$ (Δabx is right)

$\Rightarrow d \in R =$ c-side of perpendicular ac-bisector

$\cap$ b-side of $L_{ap}^{\perp} = L_{ax}$



Split R by L_{cx} into regions R_1 and R_2.

i) $d \in R_1$

Δacx is right $\Rightarrow$ Δacd is obtuse and $\|a-d\| > \|a-x\|$.

similar to case a: ..

ii) $d \in R_2$

Δacd is acute at a, b, c

c ✓

a: $cd < ad \Rightarrow cd^2 < ad^2 \quad ac^2 \Rightarrow cd^2 = ad^2 + ac^2 - 2ad \cdot ac \cdot \underbrace{\cos\alpha}_{>0} \Leftrightarrow \alpha < \frac{\pi}{2}$

d: let g, h be intersections of cd, ad with C

$$\gamma = \frac{\overset{\frown}{ag}}{2}, \quad \alpha = \frac{\overset{\frown}{ch}}{2} \Rightarrow \delta = \pi - \frac{\overset{\frown}{ag} + \overset{\frown}{ch}}{2} = \pi - \frac{\overbrace{\overset{\frown}{ac} + \overset{\frown}{gh}}_{>\pi}}{2} < \frac{\pi}{2}$$

let C' be the circle through acd with center p'

$\Rightarrow$ $p' \in$ perpendicular ac-bisector

$pd > pc$, $p'd = p'c \Rightarrow p'd < pd$, $\underbrace{p'c}_{\text{radius of } C'} > \overbrace{pc}^{\text{radius of } C}$ □

2.75 Algorithm (Smallest enclosing circle)

Input: $v_1, \dots, v_m \in \mathbb{R}^2$

Output: $C(v_1, \dots, v_m)$ // smallest circle enclosing $v_1, \dots, v_m$

Data structures: $i \in \mathbb{N}$, circles C_i, $i = 1, \dots, m-2$

1. if $m \le 2$ then output $C(v_1, \dots, v_m)$
2. $i \leftarrow 2$, choose $v_1, v_2 \in V$ randomly, $C_i \leftarrow C(v_1, v_2)$

3: If $i = m$ then goto 5

4 $i \leftarrow i+1$, choose $v_i \in V \setminus \{v_1, \dots, v_{i-1}\}$ randomly

if $v_i \in \text{conv } C_{i-1}$ then goto 3

else $C_i \leftarrow \underbrace{C^1(v_1, \dots, v_i)}_{\text{smallest enclosing circle of } v_1, \dots, v_{i-1} \text{ with } v_i \text{ on the boundary}}$, if

endif

5. output C_i

function $C^1(v_1, \dots, v_i)$

Input: $v_1, \dots, v_i \in \mathbb{R}^2$

Output: $C^1(v_1, \dots, v_i)$ // smallest circle enclosing $v_1, \dots, v_{i-1}$ with v_i on the boundary

Data structures: $j \in \mathbb{N}$, circles C^1_j, $j = 1, \dots, i-1$

1 if $i = 1$ then $C^1_i \leftarrow C(v_1)$ output C^1_i

2 $j \leftarrow 1$, choose $v_j \in V \setminus \{v_1, \dots, v_{j-1}, v_i\}$, $C_j \leftarrow C^1(v_1, \dots, v_j, v_i)$

3. if $j = i-1$ then goto 5

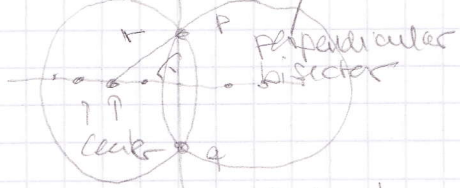
4. $j \leftarrow j+1$, choose $v_j \in V \setminus \{v_1,\dots,v_{j-1},v_i\}$ randomly

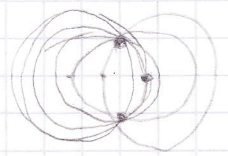
if $v_j \in \text{conv}\, C^1_{j-1}$ then goto 3

else $C^1_j \leftarrow \underbrace{C^2(v_1,\dots,v_j,v_i)}_{\text{smallest enclosing circle of } v_1,\dots,v_{j-1} \text{ with } v_j, v_i \text{ on the boundary}}$

endif

5 output C^1_j





2.7.6 Obs.: a) The center of a circle through two points p,q lies on the perpendicular pq-bisector (either left or right)

b) Let $v_1,\dots,v_m,p,q \in \mathbb{R}^2$. Then

$$C^2(v_1,\dots,v_m,p,q) = \max\Big\{ \underbrace{\max_{v_i \text{ left of } L_{pq}} C(v_i,p,q)}_{= \text{left most centered circle}},\ \underbrace{\max_{v_i \text{ right of } L_{pq}} C(v_i,p,q)}_{\text{right most centered circle}} \Big\}$$

function $C^2(v_1,\dots,v_{j-1},v_j,v_i)$

Input: $v_1,\dots,v_j,v_i \in \mathbb{R}^2$ // $m = j+1$

Output: $C^2(v_1,\dots,v_j,v_i) = C(v_1,\dots,v_j,v_i)$

Data structures: C^2_L, C^2_R, C^2 circles

1. subdivide $L \cup R \leftarrow \{v_1,\dots,v_{j-1}\}$ along perpendicular v_jv_i-bisector

2. $\left.\begin{array}{l} C^2_L \leftarrow \max_{v \in L} C(v,v_j,v_i) \\ C^2_R \leftarrow \max_{v \in R} C(v,v_j,v_i) \end{array}\right\}$ $(m-2) \times O(1)$

$C^2 \leftarrow \max C^2_L, C^2_R$ $O(1)$

3. output C^2 $O(1)$

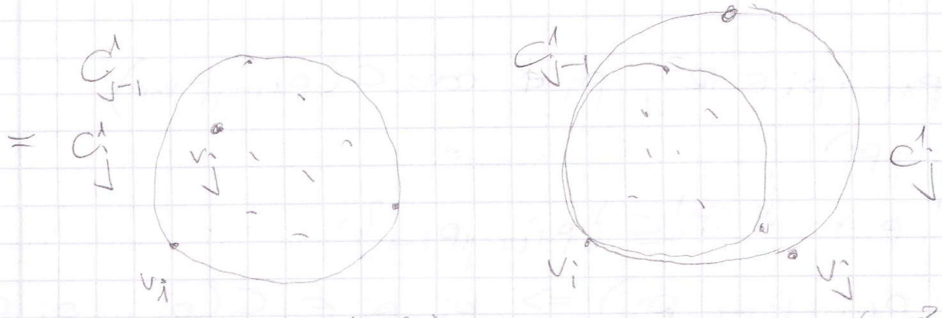
$O(m)$

2.7.7 Thm: Let $T^2(m), T^1(m), T(m)$ the expected mean time of computing $C^2(v_1,\dots,v_m), C^1(v_1,\dots,v_m), C^2(v_1,\dots,v_m)$. Then $T^2(m) = T^1(m) = T(m) = O(m)$.

Proof:

a) $T^2(m) = O(m)$

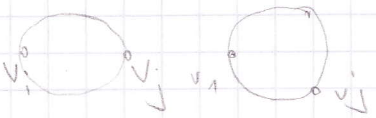
b) Consider the treatment of v_j, $j = 1,\dots,m-1$



cheap $(O(1))$ expensive $(C^2(v_1,\dots,v_j,v_i), O(j+1))$

$$P[C^1_j \text{ expensive}] = P[\underbrace{C(V' \subseteq V : |V'| = j-1)}_{\text{circle changes if } v_j \text{ is removed}} \neq C^1_j]$$

$$\le \max\left\{\frac{1}{j}, \frac{2}{j}\right\} = \frac{2}{j}$$

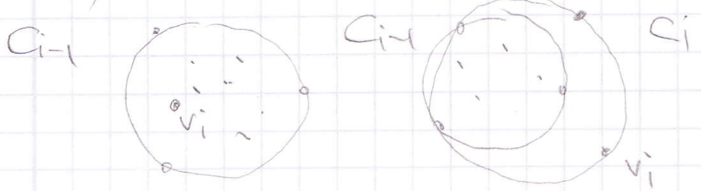


The expected time for iteration j is

$$T^1_j = \frac{j-2}{2} O(1) + \frac{2}{j} O(j) = O(1)$$

$$\Rightarrow T^1(m) = \sum_{j=1}^{m} T^1_j = O(m)$$

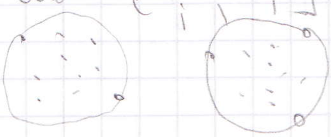
c) Consider the treatment of v_i, $i = 1, \dots, m$



cheap expensive, $O(i)$

$$P[C_i \text{ expensive}] = P[C(V' \subseteq \{v_1,\dots,v_i\}, |V'| = i-1) \neq C_i]$$

$$\le \max\left\{\frac{2}{i}, \frac{3}{i}\right\} = \frac{3}{i}$$



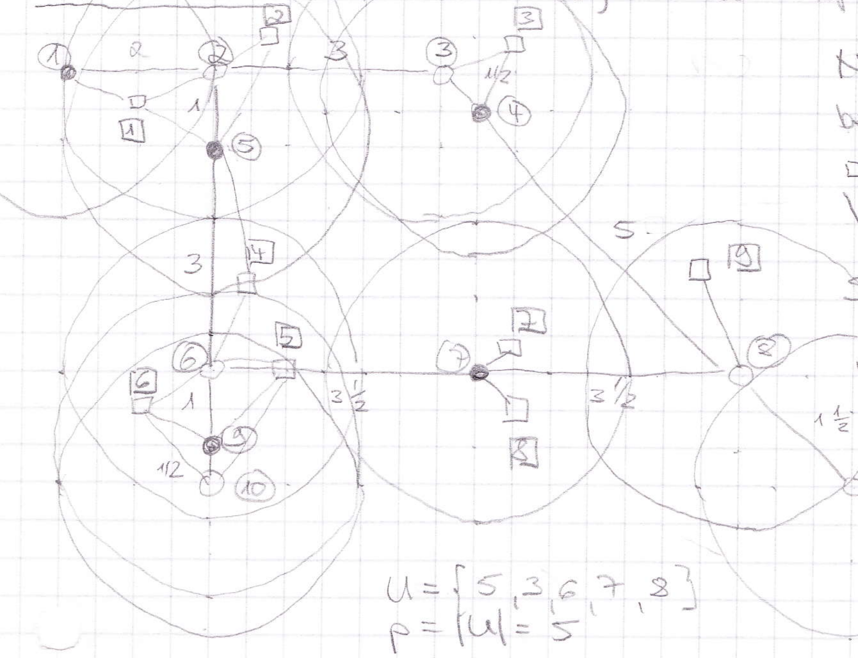
The expected time for iteration i is

$$T_i = \frac{i-3}{i} O(1) + \frac{3}{i} O(i) = O(1)$$

$$\Rightarrow T(m) = \sum_{i=1}^{m} O(1) = O(m). \quad \square$$

2.5 Stop Location Problems in Networks

2.5.1 Ex. (Schöbel [2005]): Bus stop location problem (BSLP)



$U = \{5, 3, 6, 7, 8\}$
$p = |U| = 5$

$N = (\underbrace{\overset{\circ}{S}}_{\text{breakpoints}} \cup \underbrace{\overset{\bullet}{T}}_{\text{existing stops}}, E)$ bus network

$\overset{\square}{V}$ finite set of demand points

$S, T, V \subseteq \mathbb{R}^2$ planar

$r \in \mathbb{R}_+$ covering radius

$cov_r(U) := \{v \in V : \|v, U\| \leq r\}$

$cov_r(u) := cov_r(\{u\}),\ u \in \mathbb{R}^2$ not. acc. finite cover of $U \subseteq \mathbb{R}^2$

$A^{cov}_r(U) := \left(\chi^{cov_r(u)}\right)_{u \in U}$

covering matrix assoc. with $U \subseteq \mathbb{R}^2$

$S = \{2, 3, 6, 8, 10, 11\}$

$T = \{1, 4, 5, 7, 9\}$

$V = \{1, \dots, 9\}$

$r = 2$

$$A^{cov}_2(S \cup T) = \begin{array}{c} \begin{array}{ccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \end{array} \\ \begin{pmatrix} 1 & 1 & & & 1 & & & & & & \\ & 1 & & & 1 & & & & & & \\ & & 1 & 1 & & & & & & & \\ & & & & 1 & 1 & & & & & \\ & & & & & 1 & & & 1 & 1 & \\ & & & & & 1 & & & 1 & 1 & \\ & & & & & & 1 & & & & \\ & & & & & & 1 & & & & \\ & & & & & & & 1 & & & \end{pmatrix} \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{array}$$

2.5.2. Def. (Discrete bus stop location problems):

a) (DSL) $\quad p = |U| \,/\, S \,/\, cov_r(U) = V \,/\, l_2 \,/\, p$ $\quad$ planning from scratch

b) (DSL 1) $\quad p = |U| \,/\, T \,/\, cov_r(U) = V \,/\, l_2 \,/\, p$ $\quad$ closing stops

c) (DSL 2) $\quad p = |U| \,/\, S \,/\, cov_r(U \cup T) = V \,/\, l_2 \,/\, p$ $\quad$ opening stops

d) (DSL 3) $\quad p = |U| \,/\, S \cup T \,/\, cov_r(U) = V \,/\, l_2 \,/\, p$ $\quad$ closing & opening stops

2.5.3. Obs.: (DSL i) is a special case of (DSL), $i = 1, 2, 3$.

Proof.: Homework.

2.5.4. Def. (Continuous bus stop location problem):

(CSL) $\quad p = |U| \,/\, N \,/\, cov_r(U) = V \,/\, l_2 \,/\, p$ $\quad$ planning from scratch

2.5.5. Rem. (Alternative objective functions): One can also consider

b) $p = |U| \,/\, \{V, N\} \,/\, cov_r(U) \,/\, sp \,/\, \sum_{ij \in V^2} w_{ij}$ $\quad$ travel time (more difficult)

a) $\quad$ " $\quad / \, l_2 \,/\, \sum_u w_u$ $\quad$ construction costs